Indian Statistical Institute, Bangalore Centre
Solution set of M.Math II Year, Mid-Sem Examination 2012
Fourier Analysis

1. (a) Show that

$$\sup_{\substack{f\in C_p[-\pi,\pi]\\ \|f\|_\infty\le 1}} \Big| \int_{-\pi}^{\pi} f(y)D_n(y)\,dy\Big| = \int_{-\pi}^{\pi} |D_n(y)|\,dy.$$

(b) Show that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(y)|\,dy = \infty.$$

(c) Deduce from (b) that for a dense G_δ set in $C_p[-\pi,\pi], \sup_n |(S_nf)(0)| = \infty$.

Proof. **Part (a):** Let $X = (C_p[-\pi,\pi], \|.\|_\infty), Y = \mathbb{C}$. Define the linear operator $T_n : X \to Y, n \in \mathbb{N}$ by

$$T_n(f) = S_nf(0) = \int_{-\pi}^{\pi} f(y)D_n(y)\,dy$$

Note that $\|T_nf\| \le \big[\int_{-\pi}^{\pi} |D_n(y)|\,dy\big]\|f\|_\infty$ which implies $\|T_n\| \le \int_{-\pi}^{\pi} |D_n(y)|\,dy$. Since $D_n(y)$ has a finite number of zeros, $g = \operatorname{sgn} D_n(y)$ has a finite number of jump discontinuities. Therefore by modifying it on a small neighbourhood of each discontinuities, for given $m \in \mathbb{N}$ there exists $f_m \in X$ with $\|f_m\|_\infty \le 1$ and $\sup\{|(f_m - g)(y)| : y \in [-\pi,\pi]\} < \frac{1}{m}$. Note that $|T_n(f_m)| \ge (1 - \frac{1}{m}) \int_{-\pi}^{\pi} |D_n(y)|\,dy,\ m \in \mathbb{N}$. Thus

$$\|T_n\| = \sup_{\substack{f\in C_p[-\pi,\pi]\\ \|f\|_\infty\le 1}} \Big| \int_{-\pi}^{\pi} f(y)D_n(y)\,dy\Big| = \int_{-\pi}^{\pi} |D_n(y)|\,dy.$$

Part (b): We know that

$$D_n(y) = \begin{cases} 2n+1 & \text{if } y = 0, \pm 2\pi, \pm 4\pi, \ldots; \\ \dfrac{\sin(2n+1)(\frac{y}{2})}{\sin\frac{y}{2}} & \text{otherwise.} \end{cases}$$

$$\int_{-\pi}^{\pi} |D_n(y)|\,dy \ge 2\int_{-\pi}^{\pi} \Big|\frac{\sin(2n+1)(\frac{y}{2})}{\sin\frac{y}{2}}\Big|\,dy$$

$$\geq 4 \int_0^{(2n+1)\frac{\pi}{2}} \frac{|\sin y|}{y}\, dy$$
$$\geq 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{k\pi}\, dy$$
$$\geq \frac{8}{\pi} \sum_{k=1}^{n} \frac{1}{k}$$

Thus

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(y)|\, dy = \infty.$$

Part (c): Using part (b) and (a) we have $\|T_n\| \to \infty$. Now by an application of Uniform Boundedness Principle to the family $\{T_n : X \to Y : n = 1, 2, \ldots\}$ of bounded linear maps there exists a dense G_δ set D in X such that for all $f \in D$ we have

$$\sup_n |(S_n f)(0)| = \infty.$$ □

2. Let $f : [-\pi, \pi] \to \mathbb{R}$ be continuous with $f(-\pi) = f(\pi)$ extend f to $\mathbb{R}$ by periodicity conditions. Assume that for some $\alpha > \frac{1}{2}$ we have

$$\int_{-\pi}^{\pi} |\frac{f(x+h) - f(x)}{h^\alpha}|^2 < \infty.$$

Show that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$.

Proof. Define $g_h(x) = f(x+h) - f(x),\ x \in [-\pi, \pi]$ and $h \in \mathbb{R}$. Observe that

$$\hat{g}_h(n) = (e^{inh} - 1)\hat{f}(n), \quad h \in \mathbb{R} \text{ and } |\hat{g}_h(n)| = 2|\sin\frac{nh}{2}||\hat{f}(n)|$$

By hypothesis, there exists $M \geq 0$ such that

$$\int_{-\pi}^{\pi} |g_h(x)|^2\, dx = Mh^{2\alpha}.$$

Now Parseval's identity yields

$$2\pi \sum_{-\infty}^{\infty} |\hat{g}_h(n)|^2 = Mh^{2\alpha}.$$

Then

$$2\pi \sum_{-\infty}^{\infty} 4|\sin \frac{nh}{2}|^2 |\hat{f}(n)|^2 = Mh^{2\alpha}.$$

Choose $h = \frac{\pi}{2^k}$ for arbitrary but fix $k \in \mathbb{N}$ and $2^{k-1} \leq |n| < 2^k$ which implies $\frac{\pi}{4} \leq |n|\frac{h}{2} < \frac{\pi}{2}$. Then $|\sin(\frac{|n|h}{2})|^2 = |\sin(\frac{nh}{2})|^2 \geq \frac{1}{2}$ and

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{M}{4\pi}(\frac{\pi}{2^k})^{2\alpha}.$$

By Cauchy Schwarz inequality,

$$\begin{aligned} \sum_{2^{k-1} \leq |n| < 2^k} |\hat{f}(n)| &\leq (\sum_{2^{k-1} \leq |n| < 2^k} 1^2)^{\frac{1}{2}} (\sum_{2^{k-1} \leq |n| < 2^k} |\hat{f}(n)|^2)^{\frac{1}{2}} \\ &\leq (2^k - 2^{k-1})^{\frac{1}{2}} \left[\frac{M}{4\pi}(\frac{\pi}{2^k})^{2\alpha}\right]^{\frac{1}{2}} \\ &= \frac{\sqrt{M}\pi^{\alpha - \frac{1}{2}}}{2^{\frac{3}{2}}} \frac{1}{2^{(\alpha - \frac{1}{2})k}} \end{aligned}$$

By taking summation over $k \in \mathbb{N}$ it follows that (in addition use the fact that $|\hat{f}(0)|^2 < \infty \Rightarrow |\hat{f}(0)| < \infty$),

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty. \qquad \square$$

3. Let

$$V_1 = \{f : \mathbb{R} \to \mathbb{C} : f(t) = \sum_{k=-\infty}^{\infty} a_k \chi_{[k,k+1)}(t)\}$$

with $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$ and

$$V_0 = \{g : \mathbb{R} \to \mathbb{C} : f(t) = \sum_{k=-\infty}^{\infty} b_k \chi_{[\frac{k}{2}, \frac{k+1}{2})}(t)\}$$

with $\sum_{k=-\infty}^{\infty} |b_k|^2 < \infty$. Find a relation between V_1 and V_0 like $V_0 \subset V_1$ or $V_1 \subset V_0$. Let W_0 be the orthogonal difference. Show that $W_0 =$ closed linear space$\{\phi(t-k) : k \in \mathbb{Z}\}$ for a suitable orthonormal family $\{\phi(t-k) : k \in \mathbb{Z}\}$.

Proof. Let $f \in V_1$. Then $f(t) = \sum_{k=-\infty}^{\infty} a_k \chi_{[k,k+1)}(t)$ with $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$. Define for $k \in \mathbb{N}$,

$$g(t) := \begin{cases} b_{2k} = a_k & \text{if } k \leq t < k + \frac{1}{2}; \\ b_{2k+1} = a_k & \text{if } k + \frac{1}{2} \leq t < k + 1. \end{cases}$$

Then $g \equiv f$ and $g \in V_0$. Thus $V_1 \subset V_0$.

Let $g \in W_0 = V_0 \ominus V_1$. Then $g(t) = \sum_{k=-\infty}^{\infty} b_k \chi_{[\frac{k}{2}, \frac{k+1}{2})}(t)$ with $\sum_{k=-\infty}^{\infty} |b_k|^2 < \infty$ and $g \perp V_1$. This implies $g \perp f_k = \chi_{[k,k+1)}$ for each $k \in \mathbb{Z}$. This orthogonality gives us $b_{2k} + b_{2k+1} = 0$ for each $k \in \mathbb{N}$. Then $\{e_k(t) = \frac{1}{\sqrt{2}}[\chi_{[k, \frac{2k+1}{2})} - \chi_{[\frac{2k+1}{2},\ k+1)}](t) : k \in \mathbb{Z}\}$ is a suitable orthonormal family in W_0 such that $\overline{\text{span}}\{e_k(t) = \frac{1}{\sqrt{2}}[\chi_{[k, \frac{2k+1}{2})} - \chi_{[\frac{2k+1}{2},\ k+1)}](t) : k \in \mathbb{Z}\} = W_0$.

□

4. Let $f \in L^2(\mathbb{R})$ be such that $Qf \in L^2(\mathbb{R})$ where $(Qf)(t) = tf(t)$. Let $\psi \in C^\infty(R)$ be such that $\psi(t) = 1$ for $|t| \leq 1$ and 0 for $|t| \geq 2, 0 \leq \psi \leq 1$. Define $f_n(t) = \psi(\frac{t}{n})f(t)$. Show that

$$\lim_{n \to \infty} [\|f_n - f\|^2 + \|Qf_n - Qf\|^2] = 0.$$

Proof. Note that $f_n(t) - f(t) \to 0$ as $n \to \infty$ (pointwise) and $|(f_n - f)(t)|^2 = |\psi(\frac{t}{n}) - 1|^2 |f(t)|^2 \leq |f(t)|^2$ $(0 \leq \psi \leq 1)$. Since $|f|^2 \in L^1(\mathbb{R})$, by applying dominating convergence theorem we have $\|f_n - f\|_2 \to 0$. Similarly, $(Qf_n)(t) - (Qf)(t) \to 0$ (pointwise) and $|(Qf_n - Qf)(t)|^2 = |\psi(\frac{t}{n}) - 1|^2 |t|^2 |f(t)|^2 \leq |t|^2 |f(t)|^2$ $(0 \leq \psi \leq 1)$. Since $|Qf|^2 \in L^1(\mathbb{R})$, by applying dominating convergence theorem we have $\|Qf_n - Qf\|_2 \to 0$. □

5. Let $f \in L^1(\mathbb{R})$ with $f(t) = 0$ for $|t| \geq k$ for some k, Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \int_{-k}^{k} f(t) e^{-itz} dt$. Show that g is analytic and calculate $g'(z)$.

Proof. Let $z = x + iy$ and $g(z) = u(x, y) + iv(x, y)$. From the expression of g we can write

$$u(x, y) = \int_{-k}^{k} e^{ty} f(t) \cos tx \, dt \quad \text{and} \quad v(x, y) = -\int_{-k}^{k} e^{-ty} f(t) \sin tx \, dt.$$

Then it is easy to verify that all partial derivatives u_x, u_y, v_x, v_y exist and continuous $u_x = v_y$ and $u_y = -v_x$ (Cauchy Reimann Equations) at all $(x, y) \in \mathbb{R}^2$. So g is an entire function and its derivative at any point is given by

$$\begin{aligned} g'(z) &= u_x(x,y) + iv_x(x,y) \\ &= -\int_{-k}^{k} te^{ty} f(t) \sin tx \, dt - i \int_{-k}^{k} te^{ty} f(t) \cos tx \, dt \\ &= -i \int_{-k}^{k} te^{ty} f(t) e^{-itx} \, dt \\ &= -i \int_{-k}^{k} t f(t) e^{-itz} \, dt. \end{aligned}$$

□

6. Let $A = [-2\pi, -\pi] \cup [\pi, 2\pi], e_k(t) = \dfrac{e^{-itk}}{\sqrt{2\pi}}$ for $k \in \mathbb{Z}$. Show that $\{e_k : k \in \mathbb{Z}\}$ is ONB for $L^2(A)$.

Proof. Let $k, m \in \mathbb{Z}$ with $m > k$

$$\begin{aligned} \langle \frac{e^{-itk}}{\sqrt{2\pi}}, \frac{e^{-itm}}{\sqrt{2\pi}} \rangle_{L^2(A)} &= \frac{1}{2\pi} [\int_{-2\pi}^{-\pi} e^{i(m-k)t} \, dt + \int_{\pi}^{2\pi} e^{i(m-k)t} \, dt] \\ &= \frac{1}{2\pi} [\frac{e^{it(m-k)}}{i(m-k)}]_{-2\pi}^{-\pi} + \frac{1}{2\pi} [\frac{e^{it(m-k)}}{i(m-k)}]_{\pi}^{2\pi} = 0. \end{aligned}$$

Further, it is clear that $\langle \frac{e^{-itk}}{\sqrt{2\pi}}, \frac{e^{-itk}}{\sqrt{2\pi}} \rangle_{L^2(A)} = 1.$ □

7. Let $f \in L^1[1, \infty)$. Show that there exists a_k in $[k, k+1]$ such that $f(a_k) \to 0$ as $k \to \infty$. Note that $a_k \to \infty$.

Proof. Define $c_k := \inf_{x \in [k,k+1]} |f(x)|$. For given $\epsilon > 0$,

Case 1: if there exists $k_0 \in \mathbb{N}$ such that $c_k < \epsilon$ for all $k \geq k_0$, then $c_k \to 0$. Thus there exists $a_k \in [k, k+1]$ for each $k \in \mathbb{N}$ such that $|f(a_k)| < \epsilon$ for $k \geq k_0$. Hence the result follows in this case.

Case 2: if there does not exists any $k_0 \in \mathbb{N}$ such that $c_k < \epsilon$ for all $k \geq k_0$, then $c_k \geq \epsilon$ for infinitely many $k \in \mathbb{N}$ and which contradicts to $f \in L^1[1, \infty)$. □

8. Let $f \in L^1(\mathbb{R})$, f is absolutely continuous and $f' \in L^1(\mathbb{R})$. Find a relation between $\hat{f}(s)$ and $\hat{f}'(s)$ and prove your claim.

Proof. Since f is absolutely continuous on $\mathbb{R}$, for any $t \in \mathbb{R}$ we have

$$f(t) = f(0) + \int_0^t f'(t).$$

Since $f' \in L^1(\mathbb{R})$ the following limits exist

$$\lim_{t\to\pm\infty} f(t) = f(0) + \lim_{t\to\pm\infty} \int_0^t f'(t).$$

The above limits must equal to zero because of $f \in L^1(\mathbb{R})$, otherwise, if $\lim\limits_{t\to\infty} f(t) = L \neq 0$, then $|f(t)| > \frac{L}{2}$ for all t large enough which contradicts to $f \in L^1(\mathbb{R})$. Similarly, $\lim\limits_{t\to-\infty} f(t) = 0$. Now using these limits in the following integration by parts we have

$$\begin{aligned}\hat{f}'(s) &= \int_{-\infty}^{\infty} f'(t)e^{-ist}dt = f(t)e^{-ist}\Big|_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\frac{d}{dt}e^{-ist}dt \\ &= is\int_{-\infty}^{\infty} f(t)e^{-ist}dt = is\hat{f}(s).\end{aligned}$$

□

9. Let μ be a complex valued measure on $\mathbb{R}$ with $\lambda(\mathbb{R}) < \infty$ where

$$\lambda(\mathbb{R}) = \sup\Big\{\sum_j |\mu(E_j)| : E_1, E_2, \ldots \text{ is a partition of } \mathbb{R}\Big\}.$$

(a) Show that $A = \{x : \mu(\{x\}) \neq 0\}$ is a countable subset of $\mathbb{R}$.

(b) Show that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\hat{\mu}(t)|^2 dt = \sum_A |\mu(\{x\})|^2.$$

Proof. Note that $A = \bigcup_{n=1}^{\infty}\{x : |\mu(\{x\})| > \frac{1}{n}\}$. Let us denote $B_n = \{x : |\mu(\{x\})| > \frac{1}{n}\}$ for each $n \in \mathbb{N}$. We claim that the set B_n for each $n \in \mathbb{N}$ is a finite set. Suppose if possible B_n is not a finite set, then there exists a countable infinite subset $\{x_1, x_2, \ldots\}$ of B_n. Say $E_j = \{x_j\},\ j = 1, 2, \ldots$. Since $x_j \in B_n$, we have $|\mu(E_j)| > \frac{1}{n}$ which implies

$\sum_{j=1}^{\infty} |\mu(E_j)| = \infty$ but by hypothesis $\sum_{j=1}^{\infty} |\mu(E_j)| < \infty$ (beacuse $E_1, E_2, \ldots, \mathbb{R} \setminus \bigcup_{j=1}^{\infty} E_j$ is a partition of $\mathbb{R}$). Thus B_n is a finite set for each $n \in \mathbb{N}$ and A is a countable union of finite sets. Hence A is a countable set.

Let us denote $\mu(\{x_j\}) = a_j$. Since A is countable set, we can write $\mu = \sum_k a_k \delta_{x_k}$ where δ_{x_j} is the Dirac measure for each j. By the definition of Fourier transform for finite measure we have

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{-itx}\, d\mu(x) = \sum_k a_k e^{-ix_k t}.$$

Then

$$|\hat{\mu}(t)|^2 = (\sum_k a_k e^{-ix_k t})(\sum_j \bar{a}_j e^{ix_j t}) = \sum_k |a_k|^2 + \sum_{k \neq j} a_k \overline{a_j} e^{-i(x_k - x_j)t}.$$

Integrating over the interval $[-T, T]$ and scaling by $\frac{1}{2T}$, we obtain

$$\frac{1}{2T} \int_{-T}^{T} |\mu(t)|^2\, dt = \sum_k |a_k|^2 + \sum_{k \neq j} \frac{a_k \overline{a_j}}{2T} \int_{-T}^{T} e^{-i(x_k - x_j)t} dt$$

where, by Fubini's theorem, we can interchange summation and integration since $\sum_{k \neq j} |a_k a_j| < \infty$. Observe that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i(x_k - x_j)t} dt = 0, \qquad k \neq j.$$

An application of the dominated convergence theorem, we can compute the limit as $T \to \infty$ term-by-term to obtain the desired conclusion. □

10. Find Fourier transform of

(a) $e^{-\frac{x^2}{2}}$

(b) $x^n\, e^{-\frac{x^2}{2}}$, $n = 1, 2, \ldots$

(c) $\chi_{[a,b]}$

(d) $\dfrac{x}{x^2 + 1}$. Note that this function is not in L^1.

Proof. Part (a): Since power series converge uniformly within all circles of convergence, and termwise integration is valid for uniformly convergent series,

$$\begin{aligned}\hat{f}(s) &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ixs}\, dx \\ &= \int_{-\infty}^{\infty} [\sum_{n=0}^{\infty} \frac{(-ixs)^n}{n!}] e^{-\frac{x^2}{2}}\, dx \\ &= \sum_{n=0}^{\infty} \frac{(-is)^n}{n!} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} x^n\, dx\end{aligned}$$

The integal value is zero if n is odd; if $n = 2m$, then an application of Gamma function yields the following

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} x^{2m}\, dx = \sqrt{2\pi}\frac{(2m)!}{m!2^m}.$$

Replacing these facts in the above expression it follows that

$$\hat{f}(s) = \sqrt{2\pi} e^{-\frac{s^2}{2}}.$$

Part (b): Recall that $\widehat{x^n f(x)}(s) = i^n \frac{d^n}{ds^n}\hat{f}(s)$. Then using part (a) we have $\widehat{x^n e^{-\frac{x^2}{2}}}(s) = i^n \frac{d^n}{ds^n}\sqrt{2\pi}e^{-\frac{s^2}{2}}$.

Part (c): For $s \neq 0$,

$$\begin{aligned}\widehat{\chi}_{[a,b]}(s) &= \int_{-\infty}^{\infty} \chi_{[a,b]} e^{-isx}\, dx \\ &= \int_a^b e^{-isx}\, dx = [\frac{e^{-isx}}{-is}]_a^b \\ &= \frac{e^{-isa} - e^{-isb}}{is};\end{aligned}$$

when $s = 0$, then $\widehat{\chi}_{[a,b]}(0) = b - a$.

Part (d): Define $f_n(x) = \frac{x}{x^2+1}\chi_{[n,n+1]}(x)$ for $n \in \mathbb{N}$. We know that $f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The Fourier transform of f_n is given by

$$\begin{aligned}\hat{f}_n(s) &= \int_{-\infty}^{\infty} f(x)\chi_{[-n,n]}(x) e^{-isx}\, dx \\ &= \int_{-n}^{n} f(x) e^{-isx}\, dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{-n}^{n} \frac{xe^{-isx}}{x^2+1}\,dx \\
&= \int_{-n}^{n} \frac{x(\cos sx - i\sin sx)}{x^2+1}\,dx \\
&= -2i\int_{0}^{n} \frac{x\sin(sx)}{x^2+1}\,dx.
\end{aligned}
$$

Now using the definition of Fourier transform for $L^2(\mathbb{R})$ we have the following limit pointwise a.e. on $\mathbb{R}$,

$$
\begin{aligned}
\hat{f}(s) &= \lim_{n\to\infty} \hat{f}_n(s) \\
&= -2i \lim_{n\to\infty} \int_{0}^{n} \frac{x\sin(sx)}{x^2+1}\,dx \\
&= -2i \int_{0}^{\infty} \frac{x\sin(sx)}{x^2+1}\,dx \\
&= -2i \int_{0}^{\infty} \frac{x\sin(sx)}{x^2+1}\,dx \\
&= -2i\frac{\pi e^{-s}}{2} \qquad \text{(for } s>0) \\
&= -i\pi e^{-s}.
\end{aligned}
$$

Observe that for $s<0, \hat{f}(s) = -\hat{f}(-s) = i\pi e^{s}$. Thus $\hat{f}(s) = -i\pi\ \mathrm{sgn}(s)e^{-|s|}$. □